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539 ON A LOGARITHMIC INEQUALITY*

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This paper is written by D. J. Kostić, student of the Faculty of Electrical Engineering, and it provides the answer to a problem set to students by Prof. D. S. Mitrinović.

Editorial Committee

The proofs of the following two theorems are given:

Theorem 1. *Let* $a_i>1\ (i=1, \ldots, k)$ *be real numbers and let* s *be a natural number such that* $1 \leqq s \leqq k-1$. *Further, let* $A=\{a_i;\ i=1, \ldots, k\}$. *Let various subsets of* A *with* s *elements be denoted by* A_j $\left(1 \leqq j \leqq \binom{k}{s}\right)$. *Let further*

$$p_j=\prod_{a_i \in A_j} a_i \quad \text{and} \quad q_j=\frac{\prod_{i=1}^{k} a_i}{p_j}.$$

If r *is a real number, such that* $q_j^{\frac{1+r}{2}}>p_j^{\frac{1-r}{2}}$ *for* $r>0$, *i. e.* $q_j^{\frac{1+r}{2}}<p_j^{\frac{1-r}{2}}$ *for* $r<0$, *then the following inequality*

$$\sum_{j=1}^{\binom{k}{s}} (\log_{p_j} q_j)^r \geqq \binom{k}{s}\left(\frac{k-s}{s}\right)^r \tag{1}$$

holds.

Proof. Let us consider the function $f(x)=\left(\frac{1}{x}-1\right)^r$ for $x \in (0, 1)$. At the points of interval $(0, 1)$ function f is convex if $r>2x-1$ and $r>0$, i. e. if $r<2x-1$ and $r<0$.

Let us put that $x_j=\log_\alpha p_j \left(i \leqq j \leqq \binom{k}{s}\right)$, where $\alpha=\prod_{i=1}^{k} a_i$. With the above assumptions it is directly verified that $0<x_j<1$ for all j. From the condition $q_j^{\frac{r+1}{2}}>p_j^{\frac{1-r}{2}}$ if follows $\log_\alpha p_j<\frac{r+1}{2}$. Similarly, for $r<0$, we have $\log_\alpha p_j>\frac{r+1}{2}$. Thus, in both cases the conditions for the application of JENSEN's inequality

$$f\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) \leqq \frac{1}{n}\sum_{k=1}^{n} f(y_k)$$

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are fulfilled. When applied, it follows that

$$\binom{k}{s}^{-1}\sum_{j=1}^{\binom{k}{s}}\left(\frac{1}{x_j}-1\right)^r \geqq \left(\binom{k}{s}\left(\sum_{j=1}^{\binom{k}{s}}\log p_j\right)^{-1}-1\right)^r$$

and since the equality

$$\sum_{j=1}^{\binom{k}{s}}\log_\alpha p_j=\binom{k-1}{s-1}$$

holds, inequality (1) is obtained. Thus, the theorem 1 is proved.

REMARK 1. Let us notice that inequality (1) holds even when $0<a_i<1\ (i=1, \dots, k)$ because $\log_y x=\log_{\frac{1}{y}}\frac{1}{x}$.

REMARK 2. In the case when $k=3,\ s=1,\ r=n$ (n positive integer) from inequality (1) follows the inequality

$$(\log_a bc)^n+(\log_b ca)^n+(\log_c ab)^n \geqq 3\cdot 2^n.$$

REMARK 3. Monograph **[1]**, p. 214 (inequality **3.2.46**, according to denotations in **[1]**) contains the following result:

If $n>1$ is an integer and $a_\nu>0$ for $\nu=1, \dots, n$, then

$$n\left(\sum_{\nu=1}^{n}\frac{a_\nu}{t-a_\nu}\right)^{-1}\leqq n-1\leqq n^{-1}\sum_{\nu=1}^{n}\frac{t-a_\nu}{a_\nu}, \tag{2}$$

where $t=\sum_{\nu=1}^{n}a_\nu$.

We shall show that inequality (2) could be obtained from inequality (1). Let us put $a_\nu=\log b_\nu\ (\nu=1, \dots, n)$, where $b_\nu>1$. Then on the basis of inequality (1) for $k=n,\ s=n-1$ we get

$$\sum_{\nu=1}^{n}\frac{a_\nu}{t-a_\nu}\geqq\frac{n}{n-1},$$

wherefrom the left hand side of inequality in (2) follows directly. Applying the same substitutions as in the first case, we get inequality

$$\sum_{\nu=1}^{n}\frac{t-a_\nu}{a_\nu}\geqq n(n-1)$$

from inequality (1) in the case when $s=1$ and $k=n$, wherefrom the right hand inequality in (2) follows directly.

Theorem 2. *Let $x\geqq 1$ be a real number, $1\leqq p<k$ and $1\leqq s<k$ integers, such that $p+1\equiv 0 \pmod s$, and let $a_r>0\ (r=1, \dots, k)$. Then inequality*

$$\frac{a_{r+s}+a_{r+s+1}+\dots+a_{r+s+p}}{a_r+a_{r+1}+\dots+a_{r+s-1}}\geqq k\left(\frac{p+1}{s}\right)^x \tag{3}$$

is valid, where $a_{k+r}=a_r\ (r=1, \dots, k)$.

Proof. We shall previously make a proof of (3) for $x=1$. we have directly

$$\sum_{r=1}^{k} \frac{a_{r+s}+\cdots+a_{r+s+p}}{a_r+\cdots+a_{r+s-1}}$$

$$=\sum_{r=1}^{k} \frac{a_{r+s}+\cdots+a_{r+2s-1}}{a_r+\cdots+a_{r+s-1}}$$

$$+\sum_{r=1}^{k} \frac{a_{r+2s}+\cdots+a_{r+3s-1}}{a_r+\cdots+a_{r+s-1}}+\cdots+\sum_{r=1}^{k} \frac{a_{r+p+1}+\cdots+a_{r+s+p}}{a_r+\cdots+a_{r+s-1}}$$

$$\geqq k\left(\left(\prod_{r=1}^{k} \frac{a_{r+s}+\cdots+a_{r+2s-1}}{a_r+\cdots+a_{r+s-1}}\right)^{\frac{1}{k}}+\cdots+\left(\prod_{r=1}^{k} \frac{a_{r+p+1}+\cdots+a_{r+s+p}}{a_r+\cdots+a_{r+s-1}}\right)^{\frac{1}{k}}\right)$$

$$=k\frac{p+1}{s}.$$

Applying the inequality between the mean of the order $x \geqq 1$ and the mean of the order 1, we get inequality (4) from the previously proved inequality (see [**2**], statement 1, p. 67). Putting into the obtained inequality $a_r=\log b_r$, where $b_r>1\ (r=1, \ldots, k)$, we get

$$(\log_{b_r b_{r+1} \cdots b_{r+s-1}} b_{r+s} b_{r+s+1} \cdots b_{r+s+p})^x \geqq k\left(\frac{p+1}{s}\right)^x. \tag{4}$$

Remark 4. For $x=n$ (n positive integer), $p=1$, $s=1$ and $k=3$ from (4) we obtain the inequality

$$(\log_a bc)^n+(\log_b ca)^n+(\log_c ab)^n \geqq 3 \cdot 2^n. \tag{5}$$

Remark 5. Inequality (4) holds even in the case when $0<b_r<1\ (r=1, \ldots, k)$.

Remark 6. Inequality (5) holds only when $a>1$ $b>1$, and $c>1$ (or $a<1$, $b<1$ and $c<1$). In the case that this is not fulfilled inequality (1) is shown to be not valid.

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